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SOME EXACTLY SOLVABLE PROBLEMS OF THE RADIATION OF THREE-DIMENSIONAL PERIODIC INTERNAL WAVES

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An exact solution of the problem of the generation of three-dimensional periodic internal waves in an exponentially stratified, viscous fluid is constructed in a linear approximation. The wave source is an arbitrary part of the surface of a vertical circular cylinder which moves in radial, azimuthal, and vertical directions. Solutions satisfying exact boundary conditions, describe both the beam of outgoing waves and wave boundary layers of two types: internal boundary layers, whose thickness depends on the buoyancy frequency and the geometry of the problem, and viscous boundary layers, which, as in a homogeneous fluid, are determined by kinematic viscosity and frequency. Asymptotic solutions are derived in explicit form for cylinders of large, intermediate, and small dimensions relative to the natural scales of the problem.

Introduction. Recent interest in internal waves is motivated by both the logic of development of theoretical hydrodynamics and the necessity of solving applied problems of hydrology, physical oceanography, and the dynamics of planetary atmospheres [1]. Considerable attention has been given to the theory of wave generation. In early studies of the generation of two-dimensional beams of internal waves, a horizontal oscillating cylinder was replaced by hydrodynamic sources and sinks, whose parameters were determined on the basis of the theory of a homogeneous fluid with allowance for physical features of the problem [2]. In a continuously stratified medium, a compact source radiates a wave beam whose angular position is determined by the ratio of the wave frequency ω to the buoyancy frequency N, and the transverse structure (which is unimodal for a small source and bimodal with maximum displacements at the edges of the beam for a large source) depends on the distance to the source and the ratio of the transverse dimension of the radiator a to the viscous wave scale $L_\nu = \sqrt[3]{g\nu}/N$ [3].

More recently, methods were developed to calculate the radiation of periodic internal waves by horizontal elliptic [4, 5], circular [6], or square [7] cylinders, on whose surfaces boundary conditions are satisfied approximately. Although the results of [2–6] are in qualitative agreement with data of optical and contact measurements of long-range wave fields, the calculated beams are narrower and their amplitudes are underestimated, especially for low-frequency waves [3, 6]. A more exact method for constructing solutions satisfying boundary conditions on a radiating surface in a continuously stratified, viscous fluid is proposed in [8, 9] for some special problems of the theory of generation of two-dimensional waves. The calculations take into account that a moving body gives rise to both a beam of internal waves and an internal boundary layer, whose thickness depends on the kinematic viscosity of the medium ν, the buoyancy frequency, and the angular positions of the wave beam $\theta = \arcsin(\omega/N)$ and the radiating surface φ relative to the horizontal $\delta_\varphi = \sqrt{2\nu\sin\theta/(N|\sin^2\theta - \sin^2\varphi|)}$. The general flow pattern, the transverse structure of the wave beams radiated by an oblique plate oscillating along its surface, and the regularities of decrease in amplitude with distance from the radiator agree with measurement data within the experimental error [10], although internal

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boundary flows are not visualized because of their small thickness and shading by wave perturbations, which are especially strong near the radiator. This mode of motion results in the occurrence of thin horizontal interlayers, which disturb the initial smooth density distribution at large distances from the body [10].

In most practically important cases, three-dimensional internal waves are observed because the sources generating them, as a rule, are compact in the three dimensions. The goal of the present work is to construct a solution of the linearized problem of the generation of three-dimensional internal waves and attendant perturbations by a body which performs small-amplitude oscillations. Calculations were performed for wave fields and boundary layers that form under arbitrary small displacements of a part of a vertical circular cylinder and satisfy the equations of motion and boundary conditions.

Basic Equations and Boundary Conditions. Small perturbations of an exponentially stratified, viscous, incompressible fluid are described by the linearized equations

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\nabla P + \nu\rho_0 \Delta \boldsymbol{v} - \rho g \boldsymbol{e}_z, \quad \frac{\partial \rho}{\partial t} + v_z \frac{d\rho_0}{dz} = 0, \quad \nabla \cdot \boldsymbol{v} = 0, \tag{1}$$

where P, ρ, and $\boldsymbol{v} = (v_x, v_y, v_z)$ are the variable pressure, density, and velocity of the fluid, respectively, $\rho_0(z) = \rho_{00} \exp(-z/\Lambda)$ is the unperturbed density, Λ is the buoyancy scale, ν is the kinematic viscosity, g is the free-fall acceleration in the negative direction of the z axis, and $\boldsymbol{e}_z$ is a unit vector.

To simplify intermediate calculations, we use a toroidal–poloidal potential defined by two scalar functions Ψ and Φ. The velocity is expressed in terms of this potential by the formula [11]

$$\boldsymbol{v} = \nabla \times (\boldsymbol{e}_z \Psi) + \nabla \times \nabla \times (\boldsymbol{e}_z \Phi). \tag{2}$$

In this case, the incompressibility equation of (1) is satisfied automatically.

Below, we consider only monochromatic perturbations of the form $\mathrm{e}^{-i\omega t}$ (this factor is omitted everywhere). Eliminating the pressure P from (1) and using (2), we obtain the following equations for Ψ and Φ:

$$[\omega^2 \Delta - N^2 \Delta_\perp - i\omega\nu\Delta^2]\Delta_\perp \Phi = 0, \qquad (\omega - i\nu\Delta)\Delta_\perp \Psi = 0. \tag{3}$$

Here $N^2 = g[d \ln \rho_0(z)/dz]^{-1} = g/\Lambda$ is the square of the buoyancy frequency and $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is a two-dimensional Laplacian.

The solution $\Delta_\perp \Phi = 0$ of the first equation of system (3) corresponds, by virtue of (2), to zero vertical velocity components, i.e., this motion can be described using the toroidal part of the potential Ψ, and the operator $\Delta_\perp$ can thus be eliminated from this equation. The solution $\Delta_\perp \Psi = 0$ of the second equation of system (3) describes nondissipative fluid flows. However, Eq. (1) contains the horizontal friction component $\rho_0 \nu \partial^2 \boldsymbol{v}/\partial z^2$, which is equal to zero only for motions of the form $\boldsymbol{v} = \boldsymbol{a}(x, y) + z\boldsymbol{b}(x, y)$. Such motions cannot be initiated by a source of limited dimensions, and, hence, such solutions should be ignored in the problem considered.

In view of the aforesaid, the system of governing equations becomes

$$[\omega^2 \Delta - N^2 \Delta_\perp - i\omega\nu\Delta^2]\Phi = 0, \qquad (\omega - i\nu\Delta)\Psi = 0. \tag{4}$$

Using the second equation of (4), from system (1) one obtains the expression for the pressure $P = \rho_0(i\omega + \nu\Delta)\partial\Phi/\partial z$.

The boundary conditions of the problem are the conditions of attachment on the radiating surface and decay of all perturbations at infinity. The shape of the surface and the character of its motion are chosen from the conditions that the solution must have the simplest form, the problem should remained three-dimensional, and the solution can be verified experimentally. These conditions are satisfied, for example, by a generator which is an arbitrary part of the surface of a vertical circular cylinder that performs motion of small amplitude. As examples, we consider the axisymmetric motions (vertical, radial, and torsional oscillations) and horizontal rigid-body displacements of the part of the cylinder.

General Solution of the Problem of the Generation of Motions by a Part of an Infinite Vertical Cylinder. We consider a cylindrical coordinate system (r, φ, z) in which the fluid velocity components (2) have the form

$$v_r = \frac{1}{r} \frac{\partial \Psi}{\partial \varphi} + \frac{\partial^2 \Phi}{\partial r \, \partial z}, \quad v_\varphi = -\frac{\partial \Psi}{\partial r} + \frac{1}{r} \frac{\partial^2 \Phi}{\partial \varphi \, \partial z}, \quad v_z = -\Delta_\perp \Phi.$$

In this case,

$$\Delta_\perp = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}, \qquad \Delta = \Delta_\perp + \frac{\partial^2}{\partial z^2}.$$

The motion of the perturbing surface is described by the velocity components $u_r(\varphi, z)$, $u_\varphi(\varphi, z)$, and $u_z(\varphi, z)$. Then, the boundary conditions become

$$v_r\Big|_{r=R} = u_r(\varphi, z), \qquad v_\varphi\Big|_{r=R} = u_\varphi(\varphi, z), \qquad v_z\Big|_{r=R} = u_z(\varphi, z), \tag{5}$$

where R is the radius of the cylinder.

The solution of Eqs. (4) is sought in the following form

$$\Phi = \sum_{m=-\infty}^{\infty} e^{im\varphi} \int_{-\infty}^{+\infty} [A_m(k) H^{(1)}_{|m|}(k_w r) + B_m(k) H^{(1)}_{|m|}(k_\varphi r)] e^{ikz}\, dk,$$

$$\Psi = \sum_{m=-\infty}^{\infty} e^{im\varphi} \int_{-\infty}^{+\infty} C_m(k) H^{(1)}_{|m|}(k_\nu r)\, e^{ikz}\, dk. \tag{6}$$

Here $H^{(1)}_{|m|}$ are Hankel functions of the first kind, and the wave numbers k_w, k_φ, and k_ν are defined by the formulas

$$k^2_{w,\varphi} = -k^2 - \frac{iN\cos^2\theta}{2\nu\sin\theta}\left[1 \mp \sqrt{1 - \frac{4i\nu k^2 \sin\theta}{N\cos^4\theta}}\right], \quad k^2_\nu = \frac{i\omega}{\nu} - k^2, \tag{7}$$

where the upper sign (minus) corresponds to k_w and the lower sign (plus) corresponds to k_φ. In (7), $\theta = \arcsin(\omega/N)$ is the angle between the beams of internal waves and the horizon. With allowance for the boundary conditions at infinity and the form of the Hankel functions of the first kind in (6), the roots of the dispersion equations must be such that $\mathrm{Im}\, k_w > 0$, $\mathrm{Im}\, k_\varphi > 0$, and $\mathrm{Im}\, k_\nu > 0$. For simplification of the analysis, the exact solutions (7) are expanded in a series in powers of ν. Restricting ourselves to terms with minimum powers of ν, we have

$$k_w = |k|\tan\theta + \frac{i\nu|k|^3}{2N\cos^5\theta}, \qquad k_\varphi = ik_\nu \cot\theta, \qquad k_\nu = (1+i)\sqrt{\frac{\omega}{2\nu}}. \tag{8}$$

Terms with the spectral density $A_m(k)$ in (6) describe internal waves outgoing from the cylinder and terms with $B_m(k)$ and $C_m(k)$ describe boundary layers on the cylinder surface. The expression with the coefficients $B_m(k)$ corresponds to *internal wave boundary layers*, whose thickness, as in the case of wave radiation by an oscillating plate [9], is defined by the expression $\delta_\varphi = \cot\theta\sqrt{2\nu/\omega}$ (in the notation of [9], $\varphi = \pi/2$ in the problem considered). This type of boundary layer is specific to a stratified fluid. Terms with the coefficients $C_m(k)$ define a *viscous, wave, boundary layer* of thickness $\delta_\nu = \sqrt{2\nu/\omega}$, which exists in both a homogeneous fluid and a stratified fluid (a periodic boundary layer by the nomenclature of [12]).

From the exact solutions of the dispersion equation (7) it follows that in transition to a homogeneous fluid ($N \to 0$), the expression for an internal wave boundary layer k_φ becomes the expression for a viscous wave boundary layer k_ν, which is not valid for the approximate relations (8). This means that a homogeneous fluid is not a special case of a stratified fluid but is a degenerate variant of it, which combines dissimilar structural elements. In the reverse transition from a homogeneous to a stratified fluid, the viscous wave boundary layer splits into two layers with different dependences of the thickness on the physical properties and geometry of the problem.

The difference in the physical nature of the internal and viscous wave layers is manifested in the fact that, in a viscous layer, the phase velocity is directed from the cylinder and in an internal layer, it is directed toward it.

The system of linear algebraic equations for the coefficients $A_m(k)$, $B_m(k)$, and $C_m(k)$ is obtained by substitution of solutions (6) into boundary conditions (5). Solving this system, we find

$$A_m = D_m^A/D_m, \qquad B = D_m^B/D_m, \qquad C = D_m^C/D_m, \tag{9}$$

where

$$D_m^A = iU_r k_\varphi^2 k_\nu H_{|m|}^{(1)}(k_\varphi R) H_{|m|}^{(1)'}(k_\nu R) - U_\varphi \frac{mk_\varphi^2}{R} H_{|m|}^{(1)}(k_\varphi R) H_{|m|}^{(1)}(k_\nu R)$$

$$+ kU_z \Big[k_\varphi k_\nu H_{|m|}^{(1)'}(k_\varphi R) H_{|m|}^{(1)'}(k_\nu R) - \frac{m^2}{R^2} H_{|m|}^{(1)}(k_\varphi R) H_{|m|}^{(1)}(k_\nu R) \Big],$$

$$D_m^B = -iU_r k_w^2 k_\nu H_{|m|}^{(1)}(k_w R) H_{|m|}^{(1)'}(k_\nu R) + U_\varphi \frac{mk_w^2}{R} H_{|m|}^{(1)}(k_w R) H_{|m|}^{(1)}(k_\nu R)$$

$$- kU_z \Big[k_w k_\nu H_{|m|}^{(1)'}(k_w R) H_{|m|}^{(1)'}(k_\nu R) - \frac{m^2}{R^2} H_{|m|}^{(1)}(k_w R) H_{|m|}^{(1)}(k_\nu R) \Big],$$

$$D_m^C = iU_r \frac{mk}{R} (k_w^2 - k_\varphi^2) H_{|m|}^{(1)}(k_w R) H_{|m|}^{(1)}(k_\varphi R) \tag{10}$$

$$- U_\varphi k k_w k_\varphi [k_w H_{|m|}^{(1)}(k_w R) H_{|m|}^{(1)'}(k_\varphi R) - k_\varphi H_{|m|}^{(1)'}(k_w R) H_{|m|}^{(1)}(k_\varphi R)]$$

$$+ U_z \frac{mk^2}{R} [k_w H_{|m|}^{(1)'}(k_w R) H_{|m|}^{(1)}(k_\varphi R) - k_\varphi H_{|m|}^{(1)}(k_w R) H_{|m|}^{(1)'}(k_\varphi R)],$$

$$\Delta_m = k k_w k_\varphi k_\nu [k_w H_{|m|}^{(1)}(k_w R) H_{|m|}^{(1)'}(k_\varphi R) - k_\varphi H_{|m|}^{(1)'}(k_w R) H_{|m|}^{(1)}(k_\varphi R)] H_{|m|}^{(1)'}(k_\nu R)$$

$$- \frac{km^2}{R^2} (k_w^2 - k_\varphi^2) H_{|m|}^{(1)}(k_w R) H_{|m|}^{(1)}(k_\varphi R) H_{|m|}^{(1)}(k_\nu R)$$

and the spectral source functions are

$$U_j(m,k) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_0^{2\pi} u_j(\varphi, z) \mathrm{e}^{-im\varphi}\, \mathrm{e}^{-ikz}\, d\varphi\, dz. \tag{11}$$

Here prime denotes differentiation with respect to the argument.

Relations (7) and (9)–(11) completely solve the problem of the generation of small perturbations in a viscous stratified fluid by motion of an arbitrary part of a vertical circular cylinder.

For axisymmetric motions of the source, formula (10) is significantly simplified. Since the smallness of ν leads to the inequality $|k_\varphi| \gg |k_w|$, for all R the inequality $|k_w H_0^{(1)}(k_w R) H_1^{(1)}(k_\varphi R)| \ll |k_\varphi H_1^{(1)}(k_w R) H_0^{(1)}(k_\varphi R)|$ holds. From this, we obtain

$$A_0 = \frac{1}{k_w H_1^{(1)}(k_w R)} \Big[\frac{iU_r(0,k)}{k} - \frac{U_z(0,k)}{k_\varphi} \frac{H_1^{(1)}(k_\varphi R)}{H_0^{(1)}(k_\varphi R)} \Big],$$

$$B_0 = -\frac{1}{k_\varphi^2 H_0^{(1)}(k_\varphi R)} \Big[iU_r(0,k) \frac{k_w}{k} \frac{H_0^{(1)}(k_w R)}{H_1^{(1)}(k_w R)} - U_z(0,k) \Big],$$

$$C_0 = -\frac{U_\varphi(0,k) k_w H_0^{(1)}(k_w R) H_1^{(1)}(k_\varphi R)}{k_\varphi k_\nu H_1^{(1)}(k_w R) H_0^{(1)}(k_\varphi R) H_1^{(1)}(k_\nu R)}.$$

Hence it follows that axisymmetric internal waves are generated only by radial and vertical motions of the cylinder surface. If some part of the cylinder performs torsional oscillations, only a viscous wave boundary layer is generated in a linear approximation. With allowance for nonlinearity of the form v_φ^2/r, which describes the centripetal acceleration in the Navier–Stokes equation, the viscous wave boundary layer is a source of internal waves and internal wave boundary layers, whose frequency is twice the frequency of the torsional oscillations [13].

TABLE 1

Source		Particle displacement	
dimension	type	in the beam $h(p,q)$	on the axis $h(0,q)$
$\frac{R}{L_\nu} \gg 1$	V	$\frac{iu_0}{\pi N\cos\theta}\sqrt{\frac{R}{r}}\int_0^\infty \frac{1}{k}\sin\frac{ka'}{2}f(p,q,k)\,dk$	$\frac{iu_0 a}{6\pi N}\left(\frac{2N\cos\theta}{\nu q}\right)^{1/3}\sqrt{\frac{R}{R+q\cos\theta}}\,\Gamma\left(\frac{1}{3}\right)$
$\frac{R}{\delta_\varphi} \gg 1$	F	$-\frac{u_0\tan\theta e^{-i\pi/4}}{\pi N}\sqrt{\frac{\nu}{\omega}}\int_0^\infty \sin\frac{ka'}{2}f(p,q,k)\,dk$	$-\frac{u_0 a\sin\theta e^{-i\pi/4}}{6\pi N}\left(\frac{2N\cos\theta}{\nu q}\right)^{2/3}\sqrt{\frac{\nu}{\omega}}\sqrt{\frac{R}{R+q\cos\theta}}\,\Gamma\left(\frac{2}{3}\right)$
$\frac{R}{\delta_\nu} \gg 1$	O	$\frac{iu_0\cos\varphi}{\pi N\cos\theta}\sqrt{\frac{R}{r}}\int_0^\infty \frac{1}{k}\sin\frac{ka'}{2}f(p,q,k)\,dk$	$\frac{iu_0 a\cos\varphi}{6\pi N}\left(\frac{2N\cos\theta}{\nu q}\right)^{1/3}\sqrt{\frac{R}{R+q\cos\theta}}\,\Gamma\left(\frac{1}{3}\right)$
$\frac{R}{L_\nu} \ll 1$	V	$\frac{u_0 R}{N\cos\theta}\sqrt{\frac{\sin\theta}{2\pi r}}e^{i\pi/4}\int_0^\infty \frac{1}{\sqrt{k}}\sin\frac{ka'}{2}f(p,q,k)\,dk$	$\frac{u_0 aRe^{i\pi/4}}{6q}\sqrt{\frac{\sin\theta}{\nu N}}$
$\frac{R}{\delta_\varphi} \gg 1$	F	$\frac{iu_0 R}{N\sqrt{2\pi r}}\int_0^\infty \sqrt{k}\sin\frac{ka'}{2}f(p,q,k)\,dk$	$\frac{iu_0 aR}{6N}\sqrt{\frac{\nu\cos\theta}{2\pi Nq}}\left(\frac{2N\cos\theta}{\nu q}\right)^{5/6}\Gamma\left(\frac{5}{6}\right)$
$\frac{R}{\delta_\nu} \gg 1$	O	$\frac{iu_0 R^2\cos\varphi\sin^{5/2}\theta e^{i\pi/4}}{\omega\sqrt{2\pi r}\cos\theta}\int_0^\infty \sqrt{k}\sin\frac{ka'}{2}f(p,q,k)\,dk$	$\frac{iu_0 aR^2\cos\varphi\sin\theta}{6N}\left(\frac{2N\cos\theta}{\nu q}\right)^{5/6}\sqrt{\frac{\tan\theta}{2\pi q}}\,\Gamma\left(\frac{5}{6}\right)$
$\frac{R}{L_\nu} \ll 1$	V	$\frac{u_0 R}{N\cos\theta}\sqrt{\frac{\sin\theta}{2\pi r}}e^{i\pi/4}\int_0^\infty \frac{1}{\sqrt{k}}\sin\frac{ka'}{2}f(p,q,k)\,dk$	$\frac{u_0 aRe^{i\pi/4}}{6q}\sqrt{\frac{\sin\theta}{\nu N}}$
$\frac{R}{\delta_\varphi} \ll 1$	F	$-\frac{u_0\nu\sin^{5/2}\theta\cos^{3/2}\theta e^{i\pi/4}}{\omega^2\ln(k_\varphi R/2)\sqrt{2\pi r}\cos\theta}\int_0^\infty \sqrt{k}\sin\frac{ka'}{2}f(p,q,k)\,dk$	$-\frac{u_0 a\nu e^{i\pi/4}}{6N^2\ln(k_\varphi R/2)}\sqrt{\frac{\tan\theta}{2\pi q}}\left(\frac{2N\cos\theta}{\nu q}\right)^{5/6}\Gamma\left(\frac{5}{6}\right)$
$\frac{R}{\delta_\nu} \ll 1$	O	$-\frac{iu_0 e^{i\pi/4}\cos\varphi\tan\theta}{Nk_\nu^2\ln(k_\nu R/2)}\sqrt{\frac{2\sin\theta}{\pi r}}\int_0^\infty \sqrt{k}\sin\frac{ka'}{2}f(p,q,k)\,dk$	$-\frac{u_0 a\nu e^{i\pi/4}\cos\varphi}{3N^2\ln(k_\nu R/2)}\sqrt{\frac{\tan\theta}{\pi q}}\left(\frac{2N\cos\theta}{\nu q}\right)^{5/6}\Gamma\left(\frac{5}{6}\right)$

The solutions obtained allow one to calculate wave fields and boundary flows for some types of radiators and to compare their efficiency. Technically, it is easier to implement radial and vertical displacements of the cylinder surface and also its horizontal rigid-body displacement long a straight line. The first two types of motion give rise to axisymmetric motions. In the third type, such symmetry is absent and the wave field is truly three-dimensional.

Calculation of the Fluid Flow Generated by Radiators of Three Types. Calculations were performed for the following three types of motion of a cylindrical surface.

Type 1. *A source of variable volume.* The radius of a part of a cylinder of height a changes periodically so that the nonzero component of the velocity of the cylinder surface has the form $u_r(\varphi, z) = u_0\vartheta(a/2 - |z|)$, where ϑ is a unit Heaviside function. Such a source is similar to a point monopole with nonzero current flow rate.

Type 2. *A frictional source.* A part of a cylinder of height a performs periodic vertical oscillations so that the nonzero component of the velocity of the cylinder surface has the form $u_z(\varphi, z) = u_0\vartheta(a/2 - |z|)$, and the generation of internal waves is due only to friction forces, as in the problem of [9].

Type 3. *An oscillating source.* A part of a cylinder of height a with unchanged shape performs periodic horizontal oscillations along the x axis, so that the nonzero component of the velocity of the cylinder surface has the form $u_r(\varphi, z) = u_0\vartheta(a/2 - |z|)\cos\varphi$ and $u_\varphi(\varphi, z) = -u_0\vartheta(a/2 - |z|)\sin\varphi$. Such motion is similar to the one produced by a point dipole (a combination of a source and a sink with zero instantaneous flow rate).

The unwieldy expressions (6) and (9)–(11) are considerably simpler in some extreme cases where asymptotic Hankel functions can be used. The necessary conditions are satisfied in the following cases.

1. The radius of the cylinder is rather great and exceeds the scales of all introduced perturbations (the width and wavelength of the beam of internal waves, the thicknesses of the wave boundary layers δ_ν and δ_φ), so that the inequalities $|k_w R| \gg 1$, $|k_\varphi R| \gg 1$, and $|k_\nu R| \gg 1$ hold.

2. The generator is a cylinder of intermediate radius that is much greater than the boundary-flow scales but much smaller than the characteristic scales of the internal-wave beam: $|k_\varphi R| \gg 1$, $|k_\nu R| \gg 1$ and $|k_w R| \ll 1$.

3. A thin radiator whose radius is much smaller than all scales of the problem: $|k_\varphi R| \ll 1$ and $|k_\nu R| \ll 1$.

The properties of the generated beams of internal waves and boundary layers depend on the nature of motion of the radiating surface. In the analysis of wave perturbations, for definiteness we consider a conical beam propagating upward from the source . For this, in (6) we should restrict ourselves to integration over k from $-\infty$ to 0. Integrals over k from 0 to $+\infty$ describe wave perturbations in the lower half-space. These limits of integration are due to the specificity of the field of periodic internal waves, in which the energy is transferred along the crests of waves directed from the source along the radius-vectors inclined at angle θ to the horizon, and the phase velocity is perpendicular to the crests and directed toward the central horizontal plane.

The expressions for the wave field have canonical form in a coordinate system attached to the beam (p, q) with the q axis is directed along the beam and the p axis directed along the phase velocity. The coordinate systems (p, q) and (r, z) are related by the formulas $r = R + p\sin\theta + q\cos\theta$ and $z = -p\cos\theta + q\sin\theta$.

The calculation results for the vertical displacements of the beam particles are shown in Table 1. The introduced function f is given by the formula

$$f(p, q, k) = \exp\Big(ikp - \frac{\nu k^3 q}{2N\cos\theta}\Big);$$

$a' = a\cos\theta$ is the projection of the height of the moving part of the cylinder onto the p axis. In Table 1, the letters V, F, and O denote sources of variable volume, frictional sources, and oscillating sources, respectively. Table 1 also gives expressions for the maximum displacements on the axis of a unimodal beam radiated by a source whose height a satisfies the inequality $(a^3/q) \ll \nu/(2N\cos^4\theta)$.

The spatial structures of a wave field retain the symmetry of the source. Volume and frictional sources produce axisymmetric wave fields. Oscillating sources have azimuthal field patterns proportional to $\cos\varphi$, with a maximum in the direction of displacements of the cylinder (in the transverse direction, waves are not radiated). In the fields of volume and frictional sources, particles at diametrically opposite points perform in-phase oscillations, whereas for oscillating sources, the oscillations are antiphased.

A comparison of the formulas given in Table 1 shows that the fields generated by volume and oscillating sources of large dimensions are identical with accuracy up to $\cos\varphi$ (i.e., up to the azimuthal field pattern). At large distances ($q \gg R$), the amplitudes of the fields of these sources decrease as $h(0,q) \sim q^{-5/6}$. The field of a frictional source decays faster [$h(0,q) \sim q^{-7/6}$ at large distances $q \gg R$]. With decrease in the radius, the efficiency of sources of all types decreases.

The amplitudes of waves generated by oscillating and volume sources of intermediate dimensions become equal at the distance

$$q_0 = \frac{R^3 N \sin^3\theta\cos\theta}{(2\pi)^{3/2}\nu}\,\Gamma^3\Big(\frac{5}{6}\Big) = \Big(\frac{R}{L_\nu}\Big)^3 \Lambda\sin^3\theta\cos\theta\Gamma^3\Big(\frac{5}{6}\Big),$$

and at $q < q_0$, an oscillating source is more effective, and at $q > q_0$, a volume source is more effective.

The ratio of the effectivenesses of thin volume and frictional (or oscillating) radiators tends to zero as $R\ln R$.

At the limit $R \to \infty$, the formula for the wave field of a frictional source becomes the formula obtained in [9] for wave generation by a part of a plane oscillating along the entire plane.

The structure of the boundary layer depends markedly on the nature of motion of the radiating surface. In all cases, the radiation of internal waves is accompanied by the formation *of an internal wave boundary layer* with scale δ_φ.

On the surface of volume and frictional sources, a viscous wave layer does not form. In the internal boundary layer, the displacements of particles are given by the following expressions:

— for the volume source, we have

$$h = -\frac{u_0}{\pi\omega}\,\frac{H_0^{(1)}(k_\varphi r)}{H_0^{(1)}(k_\varphi R)}\ln\Big|\frac{z+a/2}{z-a/2}\Big| \tag{12}$$

in the case of a large cylinder or

$$h = -\frac{u_0 R\tan^2\theta}{\pi\omega}\,\frac{H_0^{(1)}(k_\varphi r)}{H_0^{(1)}(k_\varphi R)}\int\limits_{-\infty}^{+\infty}\ln\,\frac{|k|R\tan\theta}{2}\,\sin\,\frac{ka}{2}\,\mathrm{e}^{ikz}\,dk \tag{13}$$

in the case of intermediate and thin cylinders;

— for frictional sources, we have

$$h = \frac{iu_z(z)}{\omega}\,\frac{H_0^{(1)}(k_\varphi r)}{H_0^{(1)}(k_\varphi R)}.$$

As the cylinder radius increases without bound, the last formula becomes the corresponding formulas of [9] for $\varphi = \pi/2$. In the case of an oscillating source, an internal boundary layer and a viscous boundary layers appear simultaneously. In the internal boundary layer there are all three velocity components present. The expression for the vertical component has the following form:

$$v_z = \frac{iu_0\cos\varphi}{\pi}\,\tan\theta\exp\left(ik_\varphi(r-R)\right)\ln\Big|\frac{z+a/2}{z-a/2}\Big| \tag{14}$$

for a large cylinder or

$$v_z = R\tan^2\theta\,\frac{H_2^{(1)}(k_\nu R)H_1^{(1)}(k_\varphi r)}{H_0^{(1)}(k_\nu R)H_1^{(1)}(k_\varphi R)}\,\frac{\partial u_r(\varphi,z)}{\partial z} \tag{15}$$

for intermediate and small cylinders. In the viscous boundary layer, motion occurs in a horizontal plane. The radial velocity component is described by the expression

$$v_r = -\frac{i}{k_\nu R}\exp\left(ik_\nu(r-R)\right)u_r(\varphi,z)$$

for a large cylinder or by the expression

$$v_r = \frac{2}{k_\nu R} \frac{H_1^{(1)}(k_\nu r)}{H_0^{(1)}(k_\nu R)} u_r(\varphi, z)$$

for intermediate and small cylinders.

The appearance of singularities at $z = \pm a/2$ in formulas (12)–(15) is due to the use of approximate expressions (8) for the wave numbers k_φ and k_ν in calculations of the corresponding integrals. When exact expressions (7) are used, the delta functions become narrow bounded peaks, and the logarithmic features disappear.

Conclusions. The problem of the radiation of three-dimensional waves by a part of an infinite vertical cylinder that performs arbitrary small displacement is solved. The calculation procedure is based on using a toroidal–poloidal potential with two scalar components.

Besides the beam of outgoing internal waves and the viscous wave (Stokes) boundary layer, whose scale is determined by the kinematic viscosity and the wave frequency ($\delta_\nu = \sqrt{2\nu/\omega}$), the exact solutions obtained (in the form of Fourier integrals) describe internal wave boundary layers, whose scales depend, in addition, on the geometry of the problem [$\delta_\varphi = f(\varphi, \theta)\delta_\nu$]. The function $f(\varphi, \theta)$ has the simplest form $f = \tan\theta$ for a vertical cylinder or $f = \sin\theta \Big/ \sqrt{|\sin^2\theta - \sin^2\varphi|}$ [9] for an oblique plane. In the particular case $\theta = \pi/4$, even with the same thicknesses of the viscous and internal wave boundary layers, complete degeneration does not occur because in the viscous layer, periodic perturbations propagate from the cylinder and in the internal layer, they propagate in the opposite direction.

In a nonlinear formulation, viscous wave boundary flow, which exists in both a homogeneous fluid and a stratified fluid, can serve as an immediate source of internal waves [13]. Using approximate solutions of the dispersion equation, one can write asymptotic approximations of the obtained expressions in explicit form and analyze the properties of the wave field component.

From the above expressions it follows that oscillating and pulsating sources of large dimensions generate similar wave fields, in which the maximum displacements near the source decrease under the same law. These fields differ in the azimuthal direction diagram: for a pulsating source, it is isotropic, and for an oscillating source, it has a figure-eight shape oriented along the direction of cylinder oscillations. Frictional sources are isotropic but less effective in all cases.

For wave fields generated by sources of intermediate dimensions, the law of attenuation depends strongly on the type and transverse dimension of the source. Isotropic fields generated by pulsating sources attenuate most slowly (in proportion to q^{-1}), and fields generated by oscillating and frictional sources damp somewhat faster (in proportion to $q^{-4/3}$). In the case of a source of large height, the maximum displacements at the center of the beam near the oscillating body can be larger than those for sources of other types. The coordinate of the point at which the wave amplitudes are identical for pulsating and oscillating sources is proportional to the cube of the ratio of the cylinder radius to the viscous wave scale and the buoyancy scale. At large distances, pulsating sources dominate. In the extreme case of a cylinder of infinite radius, the expressions for the field of a frictional source becomes the solutions [9] for perturbations generated by an oscillating vertical plate. The pattern of waves generated by an oscillating cylindrical shell [14] is in qualitative agreement with the above calculations.

Viscous and internal wave boundary layers arise not only in generation but also in reflection of three-dimensional internal waves and, as in the two-dimensional case [15], solve the problem of “critical angles.”

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